WANG'S THEOREM

KS

In this document I provide a complete and careful proof of Wang's theorem, from [Wan58], theorem **??** below.

This theorem expresses the idea that invariant connections on a (homogeneous bundle over a) homogeneous space (i.e. transitive group action) are determined by their value at a single point (like any invariant function), as one can use the group action to move around to other points. The proof is not conceptually difficult, but I wanted the details worked out somewhere. It turned out to be helpful in fleshing out the picture of homogeneous bundles, and seeing unspoken details about the way connections are defined.

1. Homogeneous Spaces (and Bundles)

Let X be a manifold and fix a point $x_0 \in X$. Let K be a Lie group which acts smoothly and transitively on X on the left. Then X is called a homogeneous space, and if H is the stabilizer of x_0, then in fact X is (equivariantly) diffeomorphic to K/H, (see [Lee12, Thm 21.18], but this is not important to us now). A different choice of x_0 will change H to a conjugate subgroup.

Now let P be a K-homogeneous principal G-bundle over X, (i.e. such that the (left) K action lifts to an action on P which commutes with the (right) G-action). Notice the group H acts on P_{x_0}, the fiber over x_0. Let p_0 be a point in this fiber. Since the G-action is free and transitive on P_{x_0}, for each $h \in H$ there is a unique element $\lambda(h) \in G$ such that

$$h.p_0 = p_0.\lambda(h) \tag{1}$$

This defines the *isotropy homomorphism*.

$$\lambda = \lambda_{p_0} : H \to G \tag{2}$$

$$p_0.\lambda(hh') = (hh').p_0 = h.p_0.\lambda(h') = (p_0.\lambda(h)).\lambda(h') = p_0.(\lambda(h)\lambda(h')) \tag{3}$$

A different choice of p_0 will change λ to a G-conjugate.

2. Invariant connections

For Lie groups K, H, and G, let $\mathfrak{k}, \mathfrak{h}, \mathfrak{g}$ denote their respective Lie algebras.

Theorem 2.1 (Wang's theorem)**.** *Let P be a K-homogeneous principal G-bundle over a manifold X with a transitive left K-action, so that $X \cong K/H$ is a homogeneous space, where H is the stabilizer of a chosen point $x_0 \in X$. Let $p_0 \in P$ be a point in the fiber over x_0, and let $\lambda : H \to G$ denote the isotropy homomorphism associated to p_0.*

Then the K-invariant connections ω on P are in one to one correspondence with linear maps $\Lambda : \mathfrak{k} \to \mathfrak{g}$ such that

(1) $\Lambda \circ \mathrm{Ad}(h) = \mathrm{Ad}(\lambda(h)) \circ \Lambda$, *for all* $h \in H$, *and*
(2) $\Lambda|_{\mathfrak{h}} = \mathrm{d}\lambda$.

The correspondence is given by, thinking of $\omega \in \Omega^1(P, \mathfrak{g})$,

$$\Lambda(v) := \omega_{p_0}(\tilde{v}) \tag{4}$$

where $\tilde{v}$ *is the vector field induced by* $v \in \mathfrak{k}$ *using the action of* K *on* P.

The curvature of such a connection can be can be computed in terms of Λ *as*

$$F_{p_0}(\tilde{v}, \tilde{w}) = [\Lambda(v), \Lambda(w)] - \Lambda[v, w]. \tag{5}$$

Remark. One can also say that Λ is an equivariant map between $\mathfrak{k}$ and $\mathfrak{g}$ as H-representations, which also satisfies (2).

Remark. Note that (**??**) determines F completely at p_0, since the map $v \mapsto \pi_* \widetilde{v}$, which is the differential of the action of K on M, is surjective onto $T_x M$ by transitivity. This is all we need, since F is a horizontal tensor.

Proof. Let ω be an invariant connection on P. We can define the linear map Λ by (**??**). For $v \in \mathfrak{h}$,

$$\omega_{p_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tv).p_0 \right) = \omega_{p_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p_0.\lambda(\exp(tv)) \right) = \omega_{p_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p_0.\exp(t\mathrm{d}\lambda(v)) \right) = \mathrm{d}\lambda(v)$$

where we used one defining property of a connection in the last step.
Now suppose $v \in \mathfrak{k}$ and $h \in H$.

$$\begin{aligned} \omega_{p_0}\left(\widetilde{\mathrm{Ad}_h(v)} \right) &= \omega_{p_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} h\exp(tv)h^{-1}.p_0 \right) = \omega_{p_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} h\exp(tv).p_0.\lambda(h^{-1}) \right) \\ &= \omega_{p_0}\left((L_h)_*(R_{\lambda(h)^{-1}})_* \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tv).p_0 \right) = (L_h)^*(R_{\lambda(h)^{-1}})^* \omega_{p_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tv).p_0 \right) \\ &= \mathrm{Ad}_{\lambda(h)}\, \omega_{p_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tv).p_0 \right) = \mathrm{Ad}_{\lambda(h)}(\Lambda(v)) \end{aligned}$$

where we used left K-invariance and right G-equivariance (the other defining property of a connection). Thus Λ satisfies the two specified properties.

Conversely let $\Lambda : \mathfrak{k} \to \mathfrak{g}$ be a linear map satisfying the above. We will first define a connection $\omega \in \Omega^1(P, \mathfrak{g})$ at p_0.

Since K acts transitively on X, the combined (transitive) action of $K \times G$ on P, defined by $(k, g).p = k.p.g^{-1}$, induces a surjection

$$s_{p_0} : \mathfrak{k} \oplus \mathfrak{g} \to T_{p_0} P \tag{6}$$

$$X \oplus Y \mapsto \widetilde{X} - \widetilde{Y}. \tag{7}$$

The kernel of this map is the tangent space to the $(K \times G)$-stabilizer of p_0, which is given by

$$\widehat{H} = \{(h, \lambda(h)) \ : \ h \in H\}$$

which has tangent space

$$\widehat{\mathfrak{h}} = \{(Z \oplus \mathrm{d}\lambda(Z)) \ : \ Z \in \mathfrak{h}\} \subset \mathfrak{k} \oplus \mathfrak{g}$$

Consider the map $\Lambda - \mathrm{id}_{\mathfrak{g}} : \mathfrak{k} \oplus \mathfrak{g} \to \mathfrak{g}$. By the second property of Λ, the kernel of this map contains $\widehat{\mathfrak{h}}$, (for $Z \in \mathfrak{h}$)

$$(\Lambda - \mathrm{id}_{\mathfrak{g}})\,(Z \oplus \mathrm{d}\lambda(Z)) = \Lambda(Z) - \mathrm{d}\lambda(Z) = 0, \tag{8}$$

so it descends to a map $T_{p_0}P \to \mathfrak{g}$. We define ω_{p_0} to be this map.

Remark. Explicitly, to evaluate $\omega_{p_0}(v)$, first find some $(X, Y) \in \mathfrak{k} \oplus \mathfrak{g}$ such that $v = \widetilde{X} - \widetilde{Y}$, and then the result is $\Lambda(X) - Y$. Notice we have minus signs just because we chose to use the map $p_0 \mapsto kp_0g^{-1}$ (which is a left action map) rather than say $p_0 \mapsto kp_0g$.

$$\begin{array}{ccc} \mathfrak{k} \oplus \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ \downarrow & \nearrow & \\ T_{p_0}P & & \end{array} \tag{9}$$

Now we want to extend to a connection ω everywhere on P by using $(K \times G)$-equivariance. Note that we want:

$$L_k^*\omega = \omega \tag{10}$$

$$\omega_{kp} = L_{k^{-1}}^*\omega_p \tag{11}$$

and

$$R_g^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega \tag{12}$$

$$\omega_{pg} = \mathrm{Ad}_{g^{-1}}\,R_{g^{-1}}^*\omega_p \tag{13}$$

$$\omega_{pg^{-1}} = \mathrm{Ad}_g\,R_g^*\omega_p \tag{14}$$

So define

$$\omega_{kp_0g^{-1}} := \mathrm{Ad}_g \circ R_g^*\,L_{k^{-1}}^*\omega_{p_0} \tag{15}$$

To show ω is indeed a K-invariant connection, it remains to show the following: (We simplify some vector notation, so that, for example if $X \in \mathfrak{k}$, we write $\widetilde{X} := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\exp(tX).p_0$ as $X.p_0$, and we write $(L_k)_*X$ as just kX.)

(1) ω is well defined.
If $p = k_1p_0g_1^{-1} = k_2p_0g_2^{-1} \in P$, then one can show that

$$h := k_1^{-1}k_2 \in H \tag{16}$$

and $p_0\,g_1^{-1}g_2 = k_1^{-1}k_2\,p_0$ implies that

$$\lambda(h) = g_1^{-1}g_2. \tag{17}$$

We will label the two possible definitions of ω_p by ω_i for $i = 1, 2$. So:

$$\omega_i := \mathrm{Ad}_{g_i}\,R_{g_i}^*\,L_{k_i^{-1}}^*\,\omega_{p_0}. \tag{18}$$

Let $v \in T_pP$. Then $k_i^{-1}vg_i \in T_{p_0}P$, so there are $X_i \in \mathfrak{k}$ and $Y_i \in \mathfrak{g}$ such that

$$k_i^{-1}vg_i = s_{p_0}(X_i, Y_i) = \widetilde{X}_i - \widetilde{Y}_i =: X_i.p_0 - p_0.Y_i \tag{19}$$

We can see that (X_1, Y_1) and (X_2, Y_2) are related through

$$k_1(X_1.p_0 - p_0.Y_1)g_1^{-1} = v = k_2(X_2.p_0 - p_0.Y_2)g_2^{-1} \tag{20}$$
$$\implies X_1.p_0 - p_0.Y_1 = k_1^{-1}k_2(X_2.p_0 - p_0.Y_2)g_2^{-1}g_1 \tag{21}$$
$$= h(X_2.p_0 - p_0.Y_2)\lambda(h)^{-1} \tag{22}$$
$$= hX_2h^{-1}.p_0 \;-\; p_0.\lambda(h)Y_2\lambda(h)^{-1} \tag{23}$$
$$= \mathrm{Ad}_h(X_2).p_0 \;-\; p_0.\,\mathrm{Ad}_{\lambda(h)}(Y_2) \tag{24}$$

We want to show that $\omega_1(v) = \omega_2(v)$.

$$\omega_2(v) = \mathrm{Ad}_{g_2}\, R_{g_2}^*\, L_{k_2^{-1}}^*\, \omega_{p_0}(v) \tag{25}$$
$$= \mathrm{Ad}_{g_2}\, \omega_{p_0}(k_2^{-1}vg_2) \tag{26}$$
$$= \mathrm{Ad}_{g_2}\, \omega_{p_0}(X_2.p_0 - p_0.Y_2) \tag{27}$$
$$= \mathrm{Ad}_{g_2}\, (\Lambda(X_2) - Y_2) \tag{28}$$
$$= \mathrm{Ad}_{g_1\cdot\lambda(h)}\, (\Lambda(X_2) - Y_2) \tag{29}$$
$$= \mathrm{Ad}_{g_1}\, \mathrm{Ad}_{\lambda(h)}\, (\Lambda(X_2) - Y_2) \tag{30}$$
$$= \mathrm{Ad}_{g_1}\, \big[\Lambda(\mathrm{Ad}_h(X_2)) - \mathrm{Ad}_{\lambda(h)}(Y_2)\big] \quad \text{by equivariance of } \Lambda \tag{31}$$
$$= \mathrm{Ad}_{g_1}\, \omega_{p_0}\, \big(\mathrm{Ad}_h(X_2).p_0 \;-\; p_0.\,\mathrm{Ad}_{\lambda(h)}(Y_2)\big) \tag{32}$$
$$= \mathrm{Ad}_{g_1}\, \omega_{p_0}\, (X_1.p_0 \;-\; p_0.Y_1) \tag{33}$$
$$= \mathrm{Ad}_{g_1}\, R_{g_1}^*\, L_{k_1^{-1}}^*\, \omega_{p_0}(v) \tag{34}$$
$$= \omega_1(v) \tag{35}$$

(2) Property 1 of connections, $\omega(\widetilde{Y}) = Y$ for $Y \in \mathfrak{g}$.
At the point p_0,

$$\omega_{p_0}(\widetilde{Y}) = \omega_{p_0}(s_{p_0}(0, -Y)) \tag{36}$$
$$= \Lambda(0) - (-Y) \tag{37}$$
$$= Y \tag{38}$$

So this is already true at p_0.
Let $p = k.p_0.g^{-1} \in P$ and $Y \in \mathfrak{g}$. Then

$$\omega_p(p.Y) = \mathrm{Ad}_g\, R_g^* L_{k^{-1}}^* \omega_{p_0}(kp_0g^{-1}Y) \tag{39}$$
$$= \mathrm{Ad}_g\, \omega_{p_0}(p_0g^{-1}Yg) \tag{40}$$
$$= \mathrm{Ad}_g\, \omega_{p_0}\, \big(p_0\, \mathrm{Ad}_{g^{-1}}(Y)\big) \tag{41}$$
$$= \mathrm{Ad}_g\, \mathrm{Ad}_{g^{-1}}(Y) \tag{42}$$
$$= Y \tag{43}$$

(3) Property 2 of connections, G-equivariance, $\mathrm{Ad}_g\, R_g^*\omega = \omega$.

For $p = k.p_0.l^{-1} \in P$ and $g \in G$,

$$\begin{aligned}
\operatorname{Ad}_g R_g^*\omega_p &= \operatorname{Ad}_g R_g^* \left[\operatorname{Ad}_l R_l^* L_k^* \omega_{p_0}\right] && (44)\\
&= \operatorname{Ad}_g \operatorname{Ad}_l R_g^* R_l^* L_k^* \omega_{p_0} && (45)\\
&= \operatorname{Ad}_g \operatorname{Ad}_l (R_l \circ R_g)^* L_k^* \omega_{p_0} && (46)\\
&= \operatorname{Ad}_{gl} R_{gl}^* L_k^* \omega_{p_0} && (47)\\
&= \omega_{k.p_0.(gl)^{-1}} && (48)\\
&= \omega_{pg^{-1}} && (49)
\end{aligned}$$

(4) K-invariance, $L_k^*\omega = \omega$.
For $p = j.p_0.g^{-1} \in P$ and $k \in K$,

$$\begin{aligned}
L_k^*\omega_p &= L_k^* \left[\operatorname{Ad}_g R_g^* L_j^* \omega_{p_0}\right] && (50)\\
&= \operatorname{Ad}_g R_g^* L_k^* L_j^* \omega_{p_0} && (51)\\
&= \operatorname{Ad}_g R_g^* L_{kj}^* \omega_{p_0} && (52)\\
&= \omega_{kj.p_0.g^{-1}} && (53)\\
&= \omega_{kp} && (54)
\end{aligned}$$

Thus, ω defined above is a K-invariant connection on P.

Next we show that these two processes ($\omega \longleftrightarrow \Lambda$) invert each other.

(1) Let $\Lambda : \mathfrak{k} \to \mathfrak{g}$ be given, and define ω as above. Then for any $X \in \mathfrak{k}$, since $s_{p_0}(X, 0) = \widetilde{X}$, it is clear from the definition that $\omega_{p_0}(\widetilde{X}) = \omega_{p_0}(s_{p_0}(X, 0)) = \Lambda(X) - 0$.
(2) Let ω be a given K-invariant connection, which defines $\Lambda : \mathfrak{k} \to \mathfrak{g}$. Let ω' be the connection form defined above. By $(K \times G)$-equivariance, ω must satisfy the formula defining ω', so if $\omega = \omega'$ at a single point, then they agree everywhere. Thus we look at the point p_0.
Let $v \in T_{p_0}P$ satisfy $v = s_{p_0}(X, Y) = Xp_0 - p_0Y$ for some $X \in \mathfrak{k}$ and $Y \in \mathfrak{g}$. Then $\omega'_{p_0}(v) = \Lambda(X) - Y$.

$$\begin{aligned}
\omega'_{p_0}(v) &= \Lambda(X) - Y && (55)\\
&= \omega_{p_0}(Xp_0) - \omega_{p_0}(p_0Y) && (56)\\
&= \omega_{p_0}(Xp_0 - p_0Y) && (57)\\
&= \omega_{p_0}(v) && (58)
\end{aligned}$$

Thus $\omega = \omega'$.

Finally we compute the curvature F of ω defined by Λ. Let $v, w \in \mathfrak{k}$. These induce vectors $\widetilde{v}_{p_0}, \widetilde{w}_{p_0}$ at $T_{p_0}P$. We will need the values of vector fields extending $\widetilde{v}_{p_0}, \widetilde{w}_{p_0}$ around p_0 in the computation, so we choose the vector fields $\widetilde{v}, \widetilde{w}$ which are also induced by the K-action, (but in the end recall that F is tensorial and does not depend on the choices of extensions).

$$\begin{aligned}
F_{p_0}(\widetilde{v}_{p_0}, \widetilde{w}_{p_0}) &= \mathrm{d}\omega(\widetilde{v}, \widetilde{w}) + \frac{1}{2}[\omega_{p_0} \wedge \omega_{p_0}](\widetilde{v}_{p_0}, \widetilde{w}_{p_0}) && (59)\\
&= \mathrm{d}\omega(\widetilde{v}, \widetilde{w}) + [\omega_{p_0}(\widetilde{v}_{p_0}), \omega_{p_0}(\widetilde{w}_{p_0})] && (60)\\
&= \widetilde{v}_{p_0} \cdot \omega(\widetilde{w}) - \widetilde{w}_{p_0} \cdot \omega(\widetilde{v}) - \omega_{p_0}\big([\widetilde{v}, \widetilde{w}]_{p_0}\big) + [\Lambda(v), \Lambda(w)] && (61)
\end{aligned}$$

Since K acts on the left, (Check:) $[\widetilde{v},\widetilde{w}] = -\widetilde{[v,w]}$, (as opposed to when a group acts on the right, in which case we have a Lie algebra homomorphism).

$$F_{p_0}(\widetilde{v}_{p_0},\widetilde{w}_{p_0}) = \Big[\mathcal{L}_{\widetilde{v}}(\omega)(\widetilde{w}_{p_0}) + \omega_{p_0}(\mathcal{L}_{\widetilde{v}}(\widetilde{w}))\Big] - \Big[\mathcal{L}_{\widetilde{w}}(\omega)(\widetilde{v}_{p_0}) + \omega_{p_0}(\mathcal{L}_{\widetilde{w}}(\widetilde{v}))\Big]$$
$$+ \omega_{p_0}\big(\widetilde{[v,w]}_{p_0}\big) + [\Lambda(v),\Lambda(w)] \tag{62}$$

Since ω is K-invariant,

$$(\mathcal{L}_{\widetilde{v}}\,\omega)|_{p_0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L^*_{\exp(tv)}\omega_{\exp(tv).p_0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \omega_{p_0} = 0. \tag{63}$$

Thus,

$$F_{p_0}(\widetilde{v}_{p_0},\widetilde{w}_{p_0}) = \omega_{p_0}([\widetilde{v},\widetilde{w}]_{p_0}) - \omega_{p_0}([\widetilde{w},\widetilde{v}]_{p_0}) + \omega_{p_0}\big(\widetilde{[v,w]}_{p_0}\big) + [\Lambda(v),\Lambda(w)] \tag{64}$$
$$= -\omega_{p_0}(\widetilde{[v,w]}_{p_0}) - \omega_{p_0}(\widetilde{[v,w]}_{p_0}) + \omega_{p_0}\big(\widetilde{[v,w]}_{p_0}\big) + [\Lambda(v),\Lambda(w)] \tag{65}$$
$$= [\Lambda(v),\Lambda(w)] - \omega_{p_0}(\widetilde{[v,w]}_{p_0}) \tag{66}$$
$$= [\Lambda(v),\Lambda(w)] - \Lambda([v,w]) \tag{67}$$

(Curvature should be the "same" everywhere by homogeneity.)

$\square$

References

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